Continuous Distribution

Uniform Distribution (Rectangular Distribution)

A random variable X is said to follow uniform distribution over an interval (a, b) if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & , a < x < b \\ 0 & , \text{otherwise.} \end{cases}$$

Here a and b (b > a) are parameters.

Derive MGF, Mean and Variance of Uniform distribution

PDF of uniform distribution is given by

$$f(x) = \frac{1}{b-a}, a < x < b.$$

$$MGF = M_X(t) = E[e^{tx}] = \int e^{tx} f(x) dx.$$

$$= \int_a^b e^{tx} \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b e^{tx} dx.$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b$$

$$= \frac{1}{b-a} \left[\frac{e^{bt} - e^{at}}{t} \right]$$

$$\boxed{\therefore M_x(t) = \frac{e^{bt} - e^{at}}{t(b-a)}}$$

Mean $= E(x) = \int x f(x) dx$

$$= \frac{1}{b-a} \int_a^b x \, dx = \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{(b+a)(b-a)}{2(b-a)}$$

Mean $= \frac{b+a}{2}$ or $\frac{a+b}{2}$

$E(x^2) = \int x^2 f(x) dx$

$$= \frac{1}{b-a} \int_a^b x^2 dx = \frac{1}{b-a} \left(\frac{x^3}{3} \right)_a^b$$

$$= \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)}$$

$$E[x^2] = \frac{b^2 + ab + a^2}{3}$$

$$\text{Variance} = E(x^2) - \left[E(x)\right]^2$$

$$= \frac{b^2+ab+a^2}{3} - \left[\frac{b+a}{2}\right]^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+\cancel{b}\,a^2}{4}.$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}.$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}.$$

$$\boxed{\text{Variance} = \frac{(b-a)^2}{12}.}$$